REOP - Session 5

Linear Programming

I/ Homework

A) Separation problem for the spanning tree polytope

MST $\min \sum_{e \in E} c(e) x_e$ s.t.

$\cancel{x_e \in \{0,1\}}$ $x_e \geq 0$ 1

$\sum_{e \in E} x_e = |V| - 1$ 2

$\sum_{e \in E[X]} x_e \leq |X| - 1 \quad \forall X \subseteq V, X \neq \emptyset, V$ 3

Separation pb: is one of the constraints broken?

$$\min_{X \subseteq V} \left(|X| - 1 - \sum_{e \in E[X]} x_e \right)$$

We consider a given solution x to $\min c^T x$ s.t. 1, 2

1. Why is it $\iff$ $\min_{\boxed{X} \subseteq V} \left(|X| + \sum_{e \notin E[X]} x_e \right)$? $\longleftarrow$ capacity of cut $\varphi(X)$

Because 2, $\sum_{e \in E} x_e = \sum_{e \in E[X]} x_e + \sum_{e \notin E[X]} x_e = |V| - 1$

The 2 objectives differ by a constant

2. $D = (U, A)$ with $U = \{o\} \cup E \cup V \cup \{d\}$ $\qquad e = (u, v)$

c) Define capacities u_a on the arcs of this flow graph

b) Given a set of vertices X, show that we can find a cut $\varphi(X)$ with capa. $\boxed{|X| + \sum_{e \notin E[X]} x_e}$

a) $u_a = 1$ for $a = (v, d)$ $\longrightarrow$ count vertices in $|X|$
$u_a = x_e$ for $a = (o, e)$ $\longrightarrow$ sum the x_e
$u_a = +\infty$ for $a = (e, v)$ $\longrightarrow$ I told you so

c) Given a minimum cut B, show that $\exists$ a cut B' of smaller capa st $B' = \varphi(X)$

b) Given $X \subset V$, we define $\varphi(X) = \{(v, d),\ v \in X\} \cup \{(o, e),\ e \notin E[X]\}$

$X = \{v_1, v_2, v_3\}$ $\qquad E[X] = \{e_1, e_2\}$

and $\varphi(X)$ is an $o-d$ cut because every path $o \to d$ goes through one of its edges

By definition, $\sum_{a \in \varphi(X)} u_a = |X| + \sum_{e \notin E[X]} x_e$

c) Given B a min capacity cut, it cannot contain edges (e, v)

We define $X = \{ v \in V, (v, d) \in B \}$

Prove that $B = \varphi(X)$

- $\forall e \in E[X], e \notin B$
- $\forall e \notin E[X], e \in B$

$B' = B$ - all edges $e \in E[X]$ st $x_e = 0$

$B' = \varphi(X)$

Let $e \in E[X]$: if $x_e = 0$ it doesn't change the capacity $u(B)$

If $x_e = 1$, we can obtain a strictly smaller cut by removing e from B

B remains a cut because a path starting with (o, e) $e \in E[X]$ must then cross an edge (v, d) with $v \in X$, and this edge belongs to B

Let $e \notin E[X]$: $e = (u, v)$ and either $u \notin X$ or $v \notin X$

if e was not part of B, then we would have an uncut path $o \to d$

so $e \in B$

II/ Modeling with linear functions

A) Polyhedra

A polyhedron is an intersection of closed half-spaces

n variables
m constraints

$$P = \bigcap_{i \in [1,m]} \{ x \in \mathbb{R}^n ; \; a_i x \leq b_i \}$$

a_i: n-dim row vector
b_i: real number

$$P = \{ x \in \mathbb{R}^n ; \; Ax \leq b \}$$

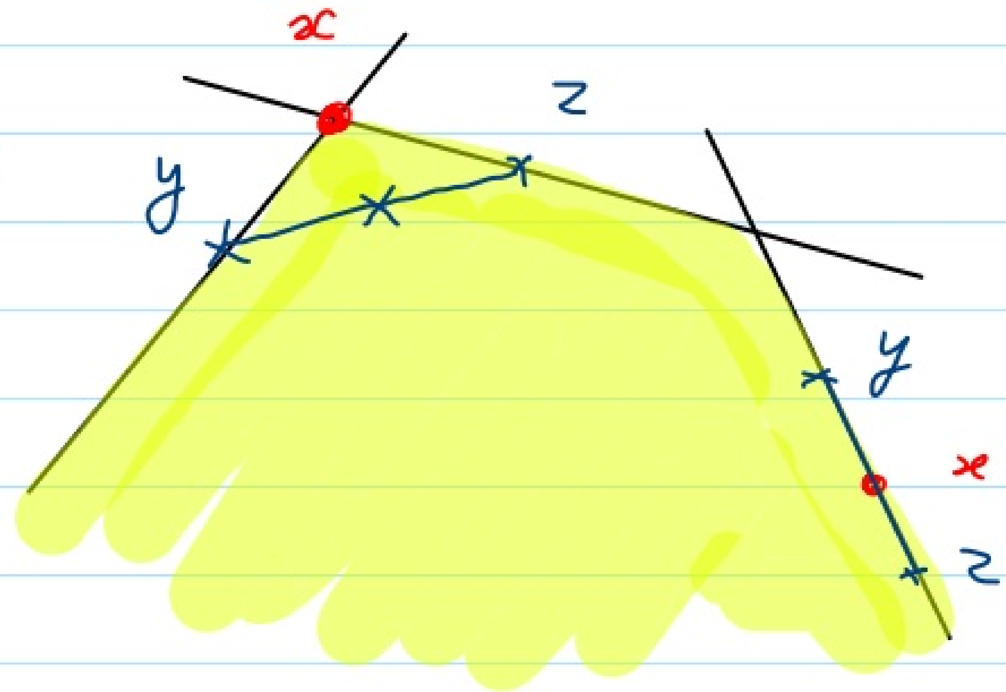
$$A = \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} \in \mathbb{R}^{m \times n}$$

$$\forall i, \; (Ax)_i \leq b_i \quad \longleftarrow \text{entrywise}$$

$$b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \in \mathbb{R}^m$$

A polytope is a bounded polyhedron

A vertex is a point $x \in P$ that cannot be written as the nontrivial mean of two other points of P



if $x = \frac{y+z}{2}$ with $\begin{matrix} y \in P \\ z \in P \end{matrix}$ then $x = y = z$

B) Linear optimization problems

A Linear Program (LP) is an optimization problem with linear objective, affine constraints

$$\min \; c^T x \text{ s.t. } Ax \leq b \qquad \text{(LP-IF)}$$

$$\Longleftrightarrow \min \; c^T x \text{ s.t. } x \in P = \{x \in \mathbb{R}^n : Ax \leq b\}$$

Standard form: $\min \; c^T x \text{ s.t. } Ax = b, \; x \geq 0 \qquad \text{(LP)}$

Ex: prove that these two forms have the same modeling power

$\Rightarrow$ Given c, A, b for (LP-IF), how to build $\tilde{c}$, $\tilde{A}$, $\tilde{b}$ and $\tilde{x}$ for (LP)

s.t. $\min \tilde{c}^T \tilde{x}$ s.t. $\tilde{A}\tilde{x} = \tilde{b}$, $\tilde{x} \geq 0$ "gives the answer" to (LP-IF)

$\min c^T x$ st $Ax \leq b$ (LP-IF)

$\min \tilde{c}^T \tilde{x}$ st $\tilde{A}\tilde{x} = \tilde{b}$, $\tilde{x} \geq 0$ (LP)

Going from (LP-IF) to (LP) : from ... to $\tilde{...}$

$Ax \leq b$ becomes $s = b - Ax$, $s \geq 0$ (slack variable)

$\min c^T x$ st $\begin{cases} Ax + s = b \\ s \geq 0 \end{cases}$ is not standard form yet (we would need $x \geq 0$)

Any number $\alpha \in \mathbb{R}$ can be written $\alpha = \beta - \gamma$ with $\beta \in \mathbb{R}_+$, $\gamma \in \mathbb{R}_+$

We decompose $x = u - v$ with $u \geq 0$, $v \geq 0$

$\min c^T(u-v)$ st $\begin{cases} A(u-v) + s = b \\ s, u, v \geq 0 \end{cases}$

$\tilde{x} = \begin{pmatrix} u \\ v \\ s \end{pmatrix}$ $\tilde{c} = \begin{pmatrix} c \\ -c \\ 0 \end{pmatrix}$ $\tilde{A} = \begin{pmatrix} A & -A & I \end{pmatrix}$ $\tilde{b} = b$

The standard form can encode any Linear Program

$\Leftarrow$: at home

We assume (without loss of generality) that rank $A = m \leq n$
$\longrightarrow$ no redundant constraints

A base $B \subset [n]$ is a subset of $|B| = m$ independent column indices of A: the submatrix $A_B = (a_{ij})_{i \in [m], j \in B}$ must be invertible (nonsingular)

Every base defines a basic solution to (LP) (in standard form)

$$x = \begin{pmatrix} x_B \\ x_N \end{pmatrix} \quad \text{with} \quad \begin{cases} x_B = A_B^{-1} b \\ x_N = 0 \end{cases}$$

$N = [n] \setminus B$

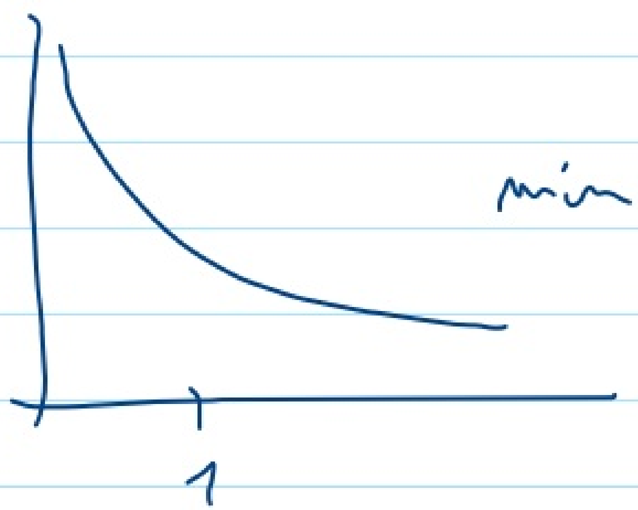
$$A x = (A_B \; A_N) \begin{pmatrix} x_B \\ x_N \end{pmatrix} = A_B x_B + A_N x_N = b + 0$$

B is called a feasible basis if $A_B^{-1} b \geq 0$ (if its basic sol is feasible)

<u>Thm</u>[FM]: x is a basic feasible solution of (LP) if & only if x is a vertex of the polyhedron $P = \{x : Ax = b, x \geq 0\}$

B) Where to find an optimum?

Thm FM : If the polyhedron P is not empty ($P \neq \emptyset$) and the objective is lower-bounded on P ($\inf_{x \in P} c^T x > -\infty$) then the problem $\min c^T x$ st $x \in P$ has (at least) one optimal solution



$\min \frac{1}{x}$ st $\underbrace{x \geq 1}_{P}$ no optimal solution

Thm 9.1 : If (LP) has an optimal solution, then at least one of the optimal solutions is basic feasible (i.e. located on a vertex of the polyhedron)

III / Solution algorithms

Naive approach: enumerate all bases / vertices & compare their costs
→ exponential runtime

It was not obvious that solving an LP is a combinatorial pb

A) Simplex algorithms

↳ Do this in a clever order. Still exponential complexity in the worst case but very efficient in practice

General idea of the simplex
1) Start from any vertex
2) Try to find one of its "neighbors" with smaller cost
↳ use reduced costs

Current solution (vertex) $\longrightarrow$ current basis $\mathcal{B}$

We can decompose any vector x as $\begin{pmatrix} x_B \\ x_N \end{pmatrix}$

$$Ax = b$$

$$A_B x_B + A_N x_N = b \text{ \& } A_B \text{ invertible}$$

$$x_B = A_B^{-1}(b - A_N x_N)$$

only $|N|$ degrees of freedom

$$\begin{aligned} c^T x &= c_B^T x_B + c_N^T x_N \\ &= c_B^T\left(A_B^{-1}(b - A_N x_N)\right) + c_N^T x_N \\ &= \underbrace{c_B^T A_B^{-1} b}_{} + \underbrace{\left(c_N^T - c_B^T A_B^{-1} A_N\right)}_{r_N^T} x_N \end{aligned}$$

r_N^T reduced cost (of nonbasic variables)

$$(c_B \; c_N) \begin{pmatrix} A_B^{-1} b \\ 0 \end{pmatrix}$$

= cost of the basic feasible solution associated with $\mathcal{B}$ $\longrightarrow$ for this solution (at the current vertex), $x_N = 0$

$$c^T x = \text{cost (current sol } \mathcal{B}) + r_N^T x_N$$

Prop 9.2: If $r_N \geq 0$ then B is an optimal basis & $x^* = \begin{pmatrix} A_B^{-1} b \\ 0 \end{pmatrix}$ is optimal

If $\exists j$, $r_j < 0$, then we can improve by increasing x_j a little bit:

↳ HW: how much?

This leads to → a new, better basis $B' \subset B \cup \{j\}$

→ a direction of unboundedness

Pivot rule: choice of j when several components of r_N are < 0

B) Other algorithms

The ellipsoid algorithm & interior point methods have worst-case polynomial complexity, but they are often less efficient in practice

Rk: The ellipsoid alg. even runs in poly time with exponential nb of constraints, as long as we can solve the separation problem efficiently

III/ Lagrangian duality (applies to general optimization pbs)

$$(P) \quad \min\ c^T x \quad \text{st} \quad \left| \begin{array}{l} Ax = b \longrightarrow \text{Lagrange multiplier } \lambda \in \mathbb{R}^{m} \to \text{nb of constraints} \\ x \geq 0 \longrightarrow \cancel{\alpha}\ \mu \in \mathbb{R}^{n}_{+} \to \text{inequality constraints} \end{array} \right.$$

Lagrangian $\mathcal{L}(x; \lambda, \mu)$ = objective + penalized constraint breaches

$$\mathcal{L}(x; \lambda, \mu) = c^T x + \sum_i \lambda_i (\underbrace{A_i x - b_i}_{>0 \,\to\, :(}) + \sum_j \mu_j (\underbrace{-x_j}_{>0 \,\to\, :(})$$

$$= c^T x + \lambda^T (Ax - b) + \mu^T (-x)$$

Primal problem (P) $\Longleftrightarrow \quad \min\limits_{x} \Big(\underbrace{\max\limits_{\lambda, \mu \geq 0} \mathcal{L}(x; \lambda, \mu)}_{= +\infty \text{ if } x \text{ doesn't satisfy the constraints}} \Big)$

Dual problem (D): $\max\limits_{\lambda, \mu \geq 0} \Big(\underbrace{\min\limits_{x} \mathcal{L}(x; \lambda, \mu)}_{d(\lambda, \mu) \text{ dual function}} \Big)$

$$\mathcal{L}(x;\lambda,\mu) = -\lambda^T b + (c^T + \lambda^T A - \mu^T)\,x$$

$$= -\lambda^T b + (c + A^T\lambda - \mu)^T x$$

$$\underbrace{\min_{\substack{x\in\mathbb{R}^n \\ \text{(no constraint)}}} \mathcal{L}(x;\lambda,\mu)}_{d(\lambda,\mu)} = \begin{cases} -\infty & \text{if the (multidimensional) slope } c + A^T\lambda - \mu \text{ has one component } \neq 0 \\ -\lambda^T b & \text{if } c + A^T\lambda - \mu = 0 \end{cases}$$

$\operatorname{argmax}_{\lambda,\mu\geq 0} d(\lambda,\mu)$ will never be in the part of the space where $d(\lambda,\mu) = -\infty$ ||

$$(D) \iff \max_{\lambda,\mu \geq 0} \; -\lambda^T b \quad \text{s.t.} \quad c + A^T\lambda - \mu = 0$$

$$\iff \max_{\lambda,\mu} \; -\lambda^T b \quad \text{s.t.} \quad \left| \begin{array}{l} c + A^T\lambda = \mu \\ \mu \geq 0 \end{array} \right.$$

$$\iff \max_{\lambda} \; -\lambda^T b \quad \text{s.t.} \quad c + A^T\lambda \geq 0$$

$$y = -\lambda$$

$$\iff \max_{y} \; + y^T b \quad \text{s.t} \quad c + A^T(-y) \geq 0$$

$$\iff \max_{y} \; b^T y \quad \text{s.t.} \quad A^T y \leq c$$

Prop 9.4

(D)

Thm 8.1: (Weak duality) For any minimization problem, we have $val(D) \leq val(P)$

Thm 9.5: (Strong duality in LP) If either (P) or (D) have a feasible solution, then $val(D) = val(P)$

Rk: these values can be infinite
What does it mean if $val(D) = val(P) = -\infty$?
primal is unbounded
dual is not feasible

Please read in the lecture notes:
- complementary slackness
- economic interpretation of dual variables

IV/ Homework for the 10^{th} of Nov.

Reduction 3-SAT to CLIQUE: I will write a solution

F. Meunier's exercise sheet on LPs (Feuille 4); ex. 6 & 11
I will write an English translation

⟶ check my website at the end of the week